Existence and Uniqueness Theorem for 1st Order ODEs

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In the previous sections, we have focused on initial value problems. This naturally raises the question: does every initial value problem admit a solution for a given ordinary differential equation, and if a solution exists, is it necessarily unique?

To illustrate this issue, consider the differential equation

$$\frac{dy}{dx} = \frac{1}{x}.$$

Since the equation is separable, we may write

$$\int 1\, dy = \int \frac{1}{x}\, dx,$$

which yields the general solution

$$y = \ln|x| + C.$$

Now suppose we are given the initial condition $y(0) = 1$. This initial condition cannot be satisfied by the general solution, since the function $\ln|x|$ is undefined at $x = 0$. Consequently, although the differential equation itself is well defined for $x \neq 0$, there exists no solution that satisfies the given initial condition.

This example demonstrates that an initial value problem may fail to have a solution if the initial condition lies outside the domain of definition of the differential equation or its solutions.

Consider the Initial Value Problem (IVP):

$$\frac{dy}{dt} = F(t, y), \quad y(t_0) = y_0 \tag{1}$$

Theorem Statement

Suppose that $F(t, y)$ and $\frac{\partial F}{\partial y}$ are continuous on a rectangle containing the point (t_0, y_0).

Then, the IVP has a **unique solution** near $(t_0, y_0) \in \mathbb{R}^2$.

Geometric Interpretation: If we consider a rectangle in the ty-plane containing (t_0, y_0), and if the continuity conditions are met within this rectangle, we are guaranteed a unique solution curve passing through (t_0, y_0) within some interval near t_0.

Linear vs Nonlinear ODEs and the Existence–Uniqueness Theorem

Example 1: Linear ODE

Problem: Find the largest interval on which the initial value problem

$$(4 - t^2)y' - 2ty = \arctan t, \quad y(-1) = 3$$

has a unique solution.

Solution:

First, rewrite the differential equation in the standard linear form $y' + p(t)y = q(t)$:

$$y' - \frac{2t}{4 - t^2}y = \frac{\arctan t}{4 - t^2}$$

Here, we identify:

$$p(t) = -\frac{2t}{4 - t^2} \quad \text{and} \quad q(t) = \frac{\arctan t}{4 - t^2}$$

According to the Existence-Uniqueness Theorem for linear equations, we need to find where $p(t)$ and $q(t)$ are continuous. The term $(4 - t^2)$ appears in the denominator, so we must ensure it is not zero:

$$4 - t^2 \neq 0 \implies t^2 \neq 4 \implies t \neq \pm 2$$

The domain is divided into three intervals:

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$$

We are given the initial condition $y(-1) = 3$. The initial point is $t_0 = -1$. We must choose the interval that contains $t_0 = -1$.

Since $-1 \in (-2, 2)$, the largest interval on which a unique solution exists is:

$$\boxed{(-2, 2)}$$

Example 2: Non-Linear ODE (Regular Case)

Problem: Is there a unique solution to the IVP:

$$y' = y^{1/3}, \quad y(0) = 1$$

near the point $(0, 1)$?

Solution:

Here, $F(t, y) = y^{1/3}$. This is a non-linear, 1st order ODE.

1. **Check continuity of $F(t, y)$:** $F(t, y) = y^{1/3}$ is continuous near $(0, 1)$.
2. **Check continuity of $\frac{\partial F}{\partial y}$:** Calculate the partial derivative:

   $$\frac{\partial F}{\partial y} = \frac{1}{3}y^{-2/3} = \frac{1}{3y^{2/3}}$$

   We evaluate this near the point $(0, 1)$. At $y = 1$:

   $$\frac{1}{3(1)^{2/3}} = \frac{1}{3}$$

   The partial derivative is defined and continuous near $y = 1$.

Conclusion: Since both F and $\frac{\partial F}{\partial y}$ are continuous on a rectangle containing $(0, 1)$, by the Existence-Uniqueness Theorem, there **is a unique solution** to the ODE near $(0, 1)$.

Example 3: Non-Linear ODE (Singular Case)

Problem 1: Is there a unique solution to the IVP:

$$y' = y^{1/3}, \quad y(0) = 0$$

near the point $(0, 0)$?

Analysis:

Check the conditions of the Existence-Uniqueness Theorem:

$$\frac{\partial F}{\partial y} = \frac{1}{3y^{2/3}}$$

At the initial point $(0,0)$, $y=0$. The derivative $\frac{\partial F}{\partial y}$ involves division by zero, so it is **not continuous** (undefined) at $y=0$.

Conclusion: The Existence-Uniqueness Theorem does **NOT** apply near $(0,0)$.

Solving the ODE:

Let's try to solve it to see what happens.

$$\frac{dy}{dt} = y^{1/3}$$

Using separation of variables ($y \neq 0$):

$$\int y^{-1/3}\,dy = \int dt$$

$$\frac{y^{2/3}}{2/3} = t + C \implies \frac{3}{2}y^{2/3} = t + C$$

Apply the initial condition $y(0)=0$:

$$\frac{3}{2}(0)^{2/3} = 0 + C \implies C = 0$$

So,

$$\frac{3}{2}y^{2/3} = t \implies y^{2/3} = \frac{2}{3}t$$

Raising both sides to the power $\frac{3}{2}$:

$$y^2 = \left(\frac{2t}{3}\right)^3 \implies y(t) = \pm\sqrt{\left(\frac{2t}{3}\right)^3}$$

This gives us two solutions for $t \geq 0$. Additionally, by inspection, the trivial solution $y(t) \equiv 0$ also satisfies $y' = y^{1/3}$ and $y(0)=0$.

Result: Thus, there are three solutions:

1. $y(t) = \sqrt{\left(\frac{2t}{3}\right)^3}$
2. $y(t) = -\sqrt{\left(\frac{2t}{3}\right)^3}$
3. $y(t) = 0$

That is, the solution exists, but is **not unique**.