Introduction to Advanced Calculus

Selim Kaan Ozsoy, Middle East Technical University

1 Historical Facts

There is a common saying that physics is applied mathematics, chemistry is applied physics, and biology is applied chemistry. In the context of this course, we find ourselves in a position where the close relationship between mathematics and physics becomes particularly apparent. Many of the ideas developed in advanced calculus emphasize how deeply physical reasoning is intertwined with mathematical structure.

This observation should not be interpreted as a claim that mathematics is superior to physics, or vice versa. Rather, it highlights the mutual dependence between the two disciplines. Mathematics provides a precise language and framework, while physics offers concrete motivation and interpretation. Neither can meaningfully advance without the other.

This perspective became especially clear when I first began studying advanced calculus. A large portion of its core concepts—particularly those belonging to vector calculus—originate from physical problems. The development of notions such as flux, circulation, and divergence was driven by the need to describe physical phenomena like fluid flow, electromagnetic fields, and conservation laws. Many of the theorems encountered in this subject are not abstract constructions, but formal expressions of physical principles.

During the eighteenth and nineteenth centuries, mathematicians such as Euler, Lagrange, Gauss, and Green developed the foundations of multivariable calculus while studying mechanics, gravitation, and electromagnetism. Their work revealed that integration over curves, surfaces, and volumes is not merely a technical extension of single-variable calculus, but a necessary language for expressing fundamental laws of nature.

2 Higher-Dimensional Calculus

In elementary calculus, much of our intuition is built on functions defined along a single direction. We integrate with respect to x or y, and the differential element dx represents a small displacement along a straight line. Geometrically, this corresponds to working on objects that are essentially one-dimensional. Even when we study areas under curves, the underlying motion of integration remains linear: we move point by point along an axis, accumulating contributions in a fixed direction.

However, many mathematical and physical phenomena are not confined to such simple settings. Quantities such as mass density, electric charge, temperature, or fluid flow are distributed over regions of space rather than along a line. To describe and analyze these situations, calculus must be extended beyond the number line and even beyond the plane. This transition marks the beginning of what is commonly referred to as advanced calculus.

A first step in this extension is the introduction of double and triple integrals. Instead of integrating along a single direction, we integrate over regions in the plane or volumes in space. The differential elements $dx\,dy$ and $dx\,dy\,dz$ no longer represent motion along a line, but rather infinitesimal pieces of area and volume. As a result, integration acquires a genuinely geometric character: double integrals naturally measure area-weighted quantities, while triple integrals capture volumetric accumulation. The increase in dimension fundamentally changes both the interpretation and the techniques of integration.