MATH 219
Spring 2025
Lecture 6

Lecture notes by Özgür Kişisel

Content: Introduction to systems of first order equations. Review of matrices.

Suggested Problems: (Boyce, Di Prima, 10th edition)

- §7.1: 19, 23

- §7.2: 1, 2, 9, 11, 16, 21, 24, 25

So far, we have explored methods of solutions and qualitative issues for a single first order differential equation in one variable. Our next goal is to consider the multi-variable version of this problem: n first order differential equations in n variables.

Just as in algebra, the essential difficulty is that the equations are almost always entangled to each other. In order to make some simplifications and decouple these equations so that each variable has its own equation, one must do a fair amount of preparatory work. In algebra, we often do this by eliminating variables, for example if we have two linear equations in two variables x and y, we try to find a combination of these equations so as to eliminate either x or y from the equation. In differential equations, the situation is similar but more intricate. The elimination step is significantly more complicated, so it pays off to develop a systematic approach to the problem.

1 Systems of first order ODE's

Suppose that $x_1, x_2, ..., x_n$ are functions of t. A set of n differential equations of the form

$$
\begin{aligned}
x_1' &= f_1(x_1, x_2, ..., x_n, t) \\
x_2' &= f_2(x_1, x_2, ..., x_n, t) \\
&\vdots \\
x_n' &= f_n(x_1, x_2, ..., x_n, t)
\end{aligned}
$$

is called a system of first order ODE's. In these equations all derivatives are with respect to the independent variable t.

Example: Consider the system

$$x' = x - 0.5xy$$
$$y' = -0.75y + 0.25xy$$

This system could be a model for describing the population dynamics of two species, so called a "predator-prey system". For example, $x(t)$ and $y(t)$ could be the populations of rabbits and foxes in a particular region. The more rabbits there are, the more they reproduce in the absence of other factors. This explains the $+$ term on the right hand side of the first equation. The more foxes and rabbits there are, the more interactions happen, namely more rabbits are devoured by foxes. Therefore it is natural to have a negative xy term on the right hand side of the first equation. Since interactions mean that the foxes find food, the coefficient of xy on the right hand side of the second equation is positive. In the absence of rabbits, the fox population dies out because of absence of food, therefore the y term on the right hand side of the second equation has a negative coefficient. Π

A solution of a system of first order ODE's as above is an n-tuple of functions $(x_1(t), x_2(t), ..., x_n(t))$. It doesn't make much sense to consider only one of the functions in the absence of others; one should consider the whole n-tuple simultaneously as a solution. An initial condition is a specification of the n values

$$x_1(t_0) = a_1, \quad x_2(t_0) = a_2, \quad ..., \quad x_n(t_0) = a_n.$$

Note that all of these values should be specified for a certain time instant t_0 and not for different instants of time. (One could consider other possibilities such as specifying different variables for different time instants, but we do not call such problems initial value problems.)

Example: Consider the initial value problem

$$x_1' = 2x_1 + x_2$$
$$x_2' = x_1 + 2x_2$$

with $x_1(0) = 5$, $x_2(0) = 1$. Verify that $x_1(t) = 3e^{3t} + 2e^t$ and $x_2(t) = 3e^{3t} - 2e^t$ is a solution of this initial value problem.

Solution: First, let us check that the two ODE's in the system are satisfied by this solution:

$$(3e^{3t} + 2e^t)' = 9e^{3t} + 2e^t = 2(3e^{3t} + 2e^t) + (3e^{3t} - 2e^t)$$
$$(3e^{3t} - 2e^t)' = 9e^{3t} - 2e^t = (3e^{3t} + 2e^t) + 2(3e^{3t} - 2e^t)$$

Next, let us check that the initial condition is satisfied:

$$x_1(0) = 3e^0 + 2e^0 = 5$$
$$x_2(0) = 3e^0 - 2e^0 = 1$$

Since all conditions of the initial value problem are satisfied, the proposed set of functions is indeed a solution.

The existence and uniqueness of solutions of a first order system of ODE's can be guaranteed in the presence of similar conditions to those that we saw in the case of a single first order ODE.

Theorem 1.1 (Existence-uniqueness theorem) Consider the system of first order ODE's

$$\begin{aligned}
x_1' &= f_1(x_1, x_2, ..., x_n, t) \\
x_2' &= f_2(x_1, x_2, ..., x_n, t) \\
&\vdots \\
x_n' &= f_n(x_1, x_2, ..., x_n, t)
\end{aligned}$$

together with the initial condition $x_1(t_0) = a_1, x_2(t_0) = a_2 ... x_n(t_0) = a_n$. Assume that each $f_i(x_1, x_2, ..., x_n, t)$ and each $\partial f_i / \partial x_j(x_1, x_2, ..., x_n, t)$ is continuous on an open rectangular box in $\mathbb{R}^{n+1}$ containing $(a_0, a_1, ..., a_n, t_0)$. Then, there is a unique solution of this initial value problem, and this solution is valid in some open interval containing t_0.

As in the case of a single equation, we will be unable to provide a proof of this theorem in these notes.

1.1 Linear systems

Recall that a first order ODE is said to be linear if it can be written in the form $y' + p(t)y = q(t)$. By analogy, we define the concept of a system of first order linear differential equations:

Definition 1.1 A system of first order ODE's is said to be linear if it can be written in the form

$$\begin{aligned}
x_1' &= a_{11}(t)x_1 + a_{12}(t)x_2 + ... + a_{1n}(t)x_n + b_1(t) \\
x_2' &= a_{21}(t)x_1 + a_{22}(t)x_2 + ... + a_{2n}(t)x_n + b_2(t) \\
&\vdots \\
x_n' &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + ... + a_{nn}(t)x_n + b_n(t)
\end{aligned}$$

for some functions $a_{ij}(t)$ and $b_i(t)$.

If furthermore $b_i(t) = 0$ for all i, then the system is said to be a homogenous system.

We will now state the existence-uniqueness theorem for first order linear systems. This theorem could be viewed as a special case of the existence-uniqueness theorem for general first order systems, which was stated above. However, we have a slightly stronger conclusion in the linear case: The solution exists throughout the interval on which the coefficient functions are continuous.

Theorem 1.2 (Existence-uniqueness theorem, linear case) Consider the linear system of ODE's

$$x_1' = a_{11}(t)x_1 + a_{12}(t)x_2 + ... + a_{1n}(t)x_n + b_1(t)$$
$$x_2' = a_{21}(t)x_1 + a_{22}(t)x_2 + ... + a_{2n}(t)x_n + b_2(t)$$
$$\vdots$$
$$x_n' = a_{n1}(t)x_1 + a_{n2}(t)x_2 + ... + a_{nn}(t)x_n + b_n(t)$$

together with the initial condition $x_1(t_0) = d_1, x_2(t_0) = d_2...x_n(t_0) = d_n$. Assume that each $a_{ij}(t)$ and $b_i(t)$ is continuous for t in an open interval (α, β). Then there is a unique solution of this initial value problem, valid over the whole interval (α, β).

The aim of this part of the course is to develop the theory of systems of first order linear ODE's and to systematically find solutions when all $a_{ij}(t)$'s are constants. Matrix algebra will be an essential tool for solving linear systems of ODE's, therefore we will spend some time reviewing basic concepts about matrices.

2 Review of Matrices

Definition 2.1 $n \times m$ matrix A (of real numbers) is a rectangular array of real numbers:

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix}$$

The numbers a_{ij} are called the entries of the matrix. The first index tells us which row the entry is on, and the second index tells us which column the entry is on. The vector $[a_{i1}, a_{i2}...a_{in}]$ is said to be the i-th row vector of the matrix A. Similarly, the vector

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ ... \\ a_{mj} \end{bmatrix}$$

is said to be the j-th column vector of the matrix A. As a shorthand notation, we will sometimes write $A = (a_{ij})$ for the whole matrix, which is meant to indicate that the ij-entry of the matrix A is a_{ij}.

2.1 Complex numbers

Recall the set $\mathbb{C}$ of complex numbers: It is the set of numbers of the form $a + bi$ where a, b are real numbers. Complex numbers are subject to the usual rules of arithmetic and $i^2 = -1$. Therefore,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i$$

The conjugate of a complex number $z = a + bi$ is $\overline{z} = a - bi$. The modulus of a complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$. It is easy to check that $z\overline{z} = |z|^2$. The only complex number with modulus zero is $z = 0$.

We can divide by any non-zero complex number. Indeed, if $z_2 \neq 0$ then

$$\frac{1}{z_2} = \frac{\overline{z}_2}{|z_2|^2}.$$

Therefore

$$\frac{z_1}{z_2} = \frac{z_1\overline{z}_2}{|z_2|^2}.$$

We can also consider matrices over complex numbers $\mathbb{C}$, by just taking the entries a_{ij} from $\mathbb{C}$.

2.2 Transpose, conjugate and adjoint

Definition 2.2 Let $A = (a_{ij})$ be an $m \times n$ matrix. Then its transpose is the $n \times m$ matrix $A^T = (a_{ji})$. The conjugate of A is $\overline{A} = (\overline{a}_{ij})$. The adjoint of A is $A^* = \overline{A}^T$.

Example: Say

$$A = \begin{bmatrix} 1 & 2 + 9i & 0 \\ -1 & 3 & 7 - 4i \end{bmatrix}$$

Then,

$$A^T = \begin{bmatrix} 1 & -1 \\ 2 + 9i & 3 \\ 0 & 7 - 4i \end{bmatrix} \quad \overline{A} = \begin{bmatrix} 1 & 2 - 9i & 0 \\ -1 & 3 & 7 + 4i \end{bmatrix} \quad A^* = \begin{bmatrix} 1 & -1 \\ 2 - 9i & 3 \\ 0 & 7 + 4i \end{bmatrix}$$

2.3 Matrix Operations

Definition 2.3 Suppose that $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices. Then their sum is the $m \times n$ matrix $A + B = (a_{ij} + b_{ij})$.

Definition 2.4 Say $A = (a_{ij})$ is an $m \times n$ matrix and c is a number. Then the scalar multiplication of A by c is the matrix $cA = (ca_{ij})$.

Subtraction can be defined by combining these two operations. Namely, $A - B = A + (-1)B$.

Definition 2.5 Say $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times k$ matrix. Then the matrix product of A and B is the $m \times k$ matrix $AB = (c_{ij})$ where

$$c_{ij} = \sum_{l=1}^{n} a_{il}b_{lj}$$

Remark 2.1 The entry c_{ij} in the definition above is equal to the dot product of the i-th row vector of A and the j-th column vector of B. The condition on the sizes of the matrices guarantees that these two vectors have the same length n.

Theorem 2.1 Matrix operations have the properties below. Let 0 denote a matrix with all entries 0, and c a constant.

- $A + B = B + A$

- $A + 0 = 0 + A = A$

- $A + (-A) = 0$

- $A + (B + C) = (A + B) + C$

- $c(A + B) = cA + cB$

- $A(BC) = (AB)C$

- $A(B + C) = AB + AC$

- $A(cB) = c(AB)$

Proof: Exercise.

Remark 2.2 In general, $AB \neq BA$ unless A and B are chosen in a special way. Matrix multiplication is not commutative.

Remark 2.3 We defined matrix operations quite mechanically and not conceptually. From this viewpoint, the rule $A(BC) = (AB)C$ (associativity of matrix multiplication) seems to be a quite miraculous fact. It is indeed an amazing fact, but it sounds more natural and less like a miracle if one uncovers the conceptual meaning of matrix multiplication: Multiplication of a matrix by a column vector essentially describes a multivariable function that takes vectors to vectors. In this setup, matrix multiplication corresponds to composition of functions. Since we know that composition of functions is an associative operation, then so is matrix multiplication.

2.4 Invertibility and Determinants

We have four basic arithmetic operations on numbers. Out of these four operations, we defined three of them for matrices. What about matrix division? First of all we have to be careful about the order of division since $AB \neq BA$. So it will be much better to write AB^{-1} or $B^{-1}A$ rather than A/B. Therefore definition of division hinges on deciding what B^{-1} means. In the case of numbers, if $a \neq 0$ then a^{-1} is the number such that $aa^{-1} = 1$. First of all we should determine which matrix plays the role of the number 1, namely the multiplicative identity element.

Definition 2.6 Suppose that I is the $n \times n$ matrix

$$I = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

namely, $I = (a_{ij})$ where $a_{ij} = 0$ if $i \neq j$ and $a_{ij} = 1$ if $i = j$. Then I is called the identity matrix of order n.

It can be easily checked that for any $n \times m$ matrix A or for any $k \times n$ matrix B,

$$IA = A$$
$$BI = B$$

Therefore the matrix I plays the role of the multiplicative identity for matrix multiplication, which is the role of the number 1 in the case of real or complex numbers.

Definition 2.7 Suppose that A is an $n \times n$ matrix. We say that the $n \times n$ matrix B is an inverse for A if

$$AB = BA = I$$

The inverse of a matrix, if exists, must be unique. Indeed, suppose that both B_1 and B_2 are inverses for A. Then $B_1 A = I$ and $AB_2 = I$. But then

$$B_1 = B_1 I = B_1 (AB_2) = (B_1 A) B_2 = IB_2 = B_2.$$

Hence there can be at most one inverse for a matrix A. Denote the inverse of A by A^{-1} if it exists.

How do we decide whether A^{-1} exists or not? In the case of numbers, this question has an easy answer: a has an inverse iff $a \neq 0$. For matrices, this is a nontrivial matter. A quick explanation can be made by using the concept of a determinant which we will discuss next.

2.5 Determinants

Let A be an $n \times n$ matrix of real or complex numbers. The determinant of A, denoted by $\det(A)$ is a number computed in terms of the entries of A. It can be computed inductively as follows:

$n = 1$: In this case $A = [a]$ and $\det(A) = a$.

$n = 2$: In this case $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det(A) = ad - bc$.

Let A_{ij} denote the matrix A with its i-th row and j-th column deleted. Therefore A_{ij} is an $(n-1) \times (n-1)$ matrix.

$n \times n$: Suppose that $A = (a_{ij})$. Then we define the determinant of A using smaller determinants, by the so called "expansion with respect to the first row":

$$\det(A) = \sum_{j=1}^{n} (-1)^{j-1} a_{1j} \det(A_{1j})$$

Example: Compute the determinant of

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 3 & 4 & 5 \\ 1 & 1 & -8 \end{bmatrix}$$

Solution:

$$\det(A) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + a_{13} \det(A_{13})$$
$$= 2 \det \left(\begin{bmatrix} 4 & 5 \\ 1 & -8 \end{bmatrix} \right) - 0 \det \left(\begin{bmatrix} 3 & 5 \\ 1 & -8 \end{bmatrix} \right) + (-2) \det \left(\begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \right)$$
$$= 2(-37) - 0 + (-2)(-1)$$
$$= -72$$

Remark 2.4 In order to avoid the awkward notation $\det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$ we write $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ for this determinant instead. A similar notation is used for larger determinants.

Now, we can state the fundamental result that relates invertibility of a matrix and the value of its determinant:

Theorem 2.2 Let A be an $n \times n$ matrix of real or complex numbers. Then A^{-1} exists if and only if $\det(A) \neq 0$.

The proof of this theorem will be omitted for the moment. Furthermore, there are alternative ways to compute the determinant of a matrix, which we will explain and use in the future lectures. Some of these will be used without proof. The right place for a complete story for matrices and determinants is a course in linear algebra.