# Method of Undetermined Coefficients

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**Recall:**

- A Higher-order linear differential equation with constant coefficients of the standard form $y^{(n)} + a_1 y^{(n-1)} + \dots a_{(n-1)} y^{(1)} = 0$ is called a homogeneous equation, where $a_1, a_2, \dots, a_{(n-1)} \in \mathbb{R}$ are constants.

- A Higher-order linear differential equation with constant coefficients of the standard form $y^{(n)} + a_1 y^{(n-1)} + \dots a_{(n-1)} y^{(1)} = g(t)$ is called a non-homogeneous equation, where $a_1, a_2, \dots, a_{(n-1)} \in \mathbb{R}$ constants, and $g(t)$ is a function of $t$.

So far, we have covered the solutions of homogeneous equations. This is significant, as we also make use of the solutions of the homogeneous form of a non-homogeneous equation while solving.
Consider a non-homogeneous equation given by

$$y^{(n)} + a_1 y^{(n-1)} + \dots a_{(n-1)} y^{(1)} = g(t).$$

Let the homogeneous solution of the equation be $y_h$ and the particular solution be $y_p$. We say that the general solution of the equation is given by

$$y = y(t) = y_h + y_p.$$

Here, it is important to underline the existence of a particular solution. A particular solution represents a solution that inherently satisfies the equation when we plug $y = y_p$. The reason for its existence is basically the term $g(t)$.
The method of undetermined coefficients to solve such equations is as follows:
Consider a non-homogeneous equation given by

$$y^{(n)} + a_1 y^{(n-1)} + \dots a_{(n-1)} y^{(1)} = g(t).$$

- Convert the equation into the homogeneous form, i.e., $y^{(n)} + a_1 y^{(n-1)} + \dots a_{(n-1)} y^{(1)} = 0$. Then, find the homogeneous solution. Call it $y_h$.

- **Guess** a particular solution, inspecting the explicit expression of $g(t)$(We will learn about this). Call it $y_p$.

- Construct the final general solution as $y = y_h + y_p$.

**An example to gain insight**: Consider the higher-order linear non-homogeneous differential equation given by

$$y'' - y = 1.$$

We wish to find a general solution using the method of undetermined coefficients. Following the strategy mentioned above, consider the homogeneous equation

$$y'' - y = 0.$$

Write down the characteristic equation by $\lambda^2 - 1 = (\lambda - 1)(\lambda + 1) = 0$ and obtain the roots $\lambda_1 = 1$ and $\lambda_2 = -1$. Therefore, the homogeneous solution is given by

$$y_h = c_1 e^t + c_2 e^{-t},$$

where $c_1$ and $c_2$ are arbitrary constants. Now, let us think for a moment. What do we need as a particular solution to get $(particular)'' - (particular) = 1$? Guess that $y_p = A$, where $A$ is a constant. Plugging in, we get

$$(A)'' - (A) = 1 \implies -A = 1 \implies A = -1$$

So, we get $A = -1$. This implies that the particular solution is given by $y_p = -1$. Hence, the general solution of the equation is given by

$$y = y(t) = y_h + y_p = c_1 e^t + c_2 e^{-t} - 1.$$

Of course, we usually do not try to derive a different **guessed particular solution** for each differential equation. We have a list where specific guessed particular solutions are noted for corresponding forcing terms; here, the forcing term stands for what we have denoted as $g(t)$.

Consider the differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t).$$

| **Forcing term** | **Guess for** $y_p(t)$ |
|---|---|
| $C$ | $A$ |
| $t^n$ | $A_n t^n + \cdots + A_1 t + A_0$ |
| $e^{at}$ | $Ae^{at}$ |
| $\sin(bt), \cos(bt)$ | $A\cos(bt) + B\sin(bt)$ |
| $e^{at} t^n$ | $e^{at}(A_n t^n + \cdots + A_0)$ |
| $e^{at}\cos(bt), e^{at}\sin(bt)$ | $e^{at}(A\cos(bt) + B\sin(bt))$ |
| Polynomial $\cdot \sin(bt)$ | Polynomial $\cdot \sin(bt)$ |
| Polynomial $\cdot e^{at}$ | Polynomial $\cdot e^{at}$ |

## Resonance Rule

Let $y_h$ be the solution of

$$a_n y^{(n)} + \cdots + a_0 y = 0.$$

If the guessed form of $y_p$ already appears in $y_h$, multiply the entire guess by $t$.
If the corresponding characteristic root has multiplicity $m$, multiply the guess by $t^m$.

$$y(t) = y_h(t) + y_p(t).$$

Example(1) Find the general solution to the non-homogeneous differential equation given as

$$y'' + 5y' + 4y = 2t + 3.$$

Solution: Following the method, we first construct the corresponding homogeneous equation. We get

$$y'' + 5y' + 4y = 0.$$

Now, the characteristic equation is given as $\lambda^2 + 5\lambda + 4 = (\lambda + 1)(\lambda + 4) = 0$. The roots are $\lambda_1 = -1$ and $\lambda_2 = -4$. Hence, the homogeneous solution is given by

$$y_h = c_1 e^{-t} + c_2 e^{-4t},$$

where $c_1, c_2$ are arbitrary constants.
Secondly, we wish to guess a particular solution. Let us inspect the forcing term, i.e., $2t + 3$. We see that the forcing term involves a polynomial of degree 1. So, by the table above, we guess the particular solution of the differential equation as $y_p = At + B$., where $A$ and $B$ are constants. We want to plug in the particular solution to the equation in order to find the constants $A$ and $B$. Notice that

$$\begin{aligned} y_p' &= A \\ y_p'' &= 0 \end{aligned}$$

and so, plugging in,

$$\begin{aligned} &0 + 5(A) + 4(At + B) = 2t + 3 \\ &5A + 4At + 4B = 2t + 3 \\ &\begin{cases} 4A = 2 \\ 5A + 4B = 3 \end{cases} \end{aligned}$$

Solving the equations, we obtain $A = 1/2$ and $B = 1/8$. Therefore, the particular solution is $y_p = \frac{t}{2} + \frac{1}{8}$. Hence, finally, the general solution of the differential equation is given by

$$y = y_h + y_p = c_1 e^{-t} + c_2 e^{-4t} + \frac{t}{2} + \frac{1}{8}.$$

Example(2) Solve the initial value problem

$$y'' - 4y = 8e^{2t}, \quad y(0) = y'(0) = 1.$$

Solution: We first construct the corresponding homogeneous equation as

$$y'' - 4y = 0.$$

The characteristic equation is given by $\lambda^2 - 4 = (\lambda - 2)(\lambda + 2) = 0$. The roots are clearly $\lambda_1 = 2$ and $\lambda_2 = -2$. Therefore, the homogeneous solution is given by $y_h = c_1 e^{2t} + c_2 e^{-2t}$, where $c_1, c_2$ are arbitrary constants. Now, we wish to guess a particular solution. Consider $Ae^{2t}$. However, we observe that the homogeneous solution already includes a term in which $e^{2t}$ occurs, namely $c_1 e^{2t}$. Therefore, we must multiply $Ae^{2t}$ by $t$ and guess the particular solution as $y_p = t\left(Ae^{2t}\right)$, where $A$ is a constant. Now, we get

$$\begin{aligned} y_p' &= Ae^{2t} + t\left(2Ae^{2t}\right) \\ y_p'' &= 2Ae^{2t} + 2Ae^{2t} + t\left(4Ae^{2t}\right) = \left(4Ae^{2t}\right)(t+1) \end{aligned}$$

and then, plugging in the particular solution,

$$\begin{aligned} \left(4Ae^{2t}\right)(t+1) - 4\left(tAe^{2t}\right) &= 4Ae^{2t} \\ &= 8e^{2t}. \end{aligned}$$

Hence, $A = 2$, and $y_p = 2te^{2t}$. Finally, the general solution is given by

$$y(t) = y_h + y_p = y_h + y_p = c_1 e^{2t} + c_2 e^{-2t} + 2te^{2t}.$$

Now, we will consider the initial conditions $y(0) = y'(0) = 1$ and find the solution of the IVP by determining $c_1$ and $c_2$. Consider

$$\begin{aligned} &y(0) = c_1 + c_2 = 1 \\ &y'(0) = 2c_1 - 2c_2 + 2 = 1 \implies y'(0) = c_1 - c_2 = -\frac{1}{2}. \end{aligned}$$

Solving, we obtain $c_1 = \frac{1}{4}$ and $c_2 = \frac{3}{4}$. Hence, the solution of the IVP is given by

$$y = \frac{1}{4}e^{2t} + \frac{3}{4}e^{-2t} + 2te^{2t}.$$