ATTENTION

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MATH 219
Spring 2025
Lecture 12

Lecture notes by Özgür Kişisel

Content: Nonhomogenous linear systems (variation of parameters only).

Suggested Problems: (Boyce, Di Prima, 10th edition)

- §7.9: 2, 6, 7, 9, 11, 13

1 Variation of parameters

In the previous lecture, we outlined a method to solve any constant coefficient homogenous linear system. Suppose now that we have a nonhomogenous linear system:

$$x' = A(t)x + b$$

Recall that a fundamental matrix $\Psi(t)$ is any matrix satisfying

$$\frac{d\Psi}{dt} = A\Psi$$
$$\det(\Psi) \neq 0$$

Provided that we can find such a matrix $\Psi(t)$, we can write down all solutions of the homogenous system $x' = Ax$ as

$$x = \Psi(t)c$$

where c is a vector of constants. In particular if A is a constant matrix, then e^{At} or Pe^{Jt} that were found in the previous lecture are fundamental matrices.

We will use a method called variation of parameters in order to solve the nonhomogenous system. The idea of variation of parameters is to replace the constant vector c in the formula $x = \Psi(t)c$ by a nonconstant vector $v(t)$ and hope that we can extract a solution of the nonhomogenous system of the form $\Psi(t)v(t)$. In fact,

Theorem 1.1 All solutions of $x' = Ax + b$ are of the form $x = \Psi(t)v(t)$ where $v(t) = \int \Psi^{-1}(t)b(t)dt$.

Proof: Plug $x = \Psi(t)v$ into the differential equation and use product rule to differentiate:

$$\begin{aligned} x' &= \frac{d\Psi}{dt}v + \Psi\frac{dv}{dt} \\ &= A\Psi v + \Psi\frac{dv}{dt} \end{aligned}$$

We want the right hand side of this equation to be equal to $Ax + b$ namely to $A\Psi v + b$. This equality holds if and only if

$$\begin{aligned} \Psi\frac{dv}{dt} &= b \\ \frac{dv}{dt} &= \Psi^{-1}b \\ v &= \int \Psi^{-1}bdt \end{aligned}$$

Therefore the expression $x = \Psi \int \Psi^{-1}bdt$ in the statement is really a solution. How can we be sure that there are no other solutions? We can write the indefinite integral above as $\int = \int_0^t + c$ where c is a vector of constants. Then the solutions obtained above are of the form $x = \Psi c + \Psi \int_0^t \Psi^{-1}(\tau)b(\tau)d\tau$. Then

$$x_p = \Psi \int_0^t \Psi^{-1}(\tau)b(\tau)d\tau$$

is a particular solution of the nonhomogenous system. If x is any other solution, then by the principle of superposition $x - x_p$ must be a solution of the corresponding homogenous system, therefore it must be of the form Ψc. This proves the claim.

Example: Solve the system $x' = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} e^{2t} \\ t \end{bmatrix}$

Solution:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda).$$

Therefore the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$. Let us find the eigenvectors for λ_1:

$$\left[\begin{array}{cc|c} 0 & 3 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{R_1/3 \to R_1} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So the eigenvectors are of the form $k\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Next, let us find the eigenvectors for λ_2. The matrix $\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$ is already in row echelon form. So the eigenvectors are of the form $k\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

Therefore we can write down two linearly independent solutions $x^{(1)} = \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix}$ and $x^{(2)} = \begin{bmatrix} -3e^t \\ e^t \end{bmatrix}$. So a fundamental matrix is

$$\Psi(t) = \begin{bmatrix} e^{2t} & -3e^t \\ 0 & e^t \end{bmatrix}$$

Its inverse can be easily computed to be $\Psi^{-1} = \begin{bmatrix} e^{-2t} & 3e^{-2t} \\ 0 & e^{-t} \end{bmatrix}$.

Now use the formula $v = \int \Psi^{-1} b dt$;

$$\begin{aligned} v &= \int \begin{bmatrix} e^{-2t} & 3e^{-2t} \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} e^{2t} \\ t \end{bmatrix} dt \\ &= \begin{bmatrix} \int (1 + 3te^{-2t}) dt \\ \int te^{-t} dt \end{bmatrix} \\ &= \begin{bmatrix} t - \frac{3}{2} te^{-2t} - \frac{3}{4} e^{-2t} + c_1 \\ -te^{-t} - e^{-t} + c_2 \end{bmatrix} \end{aligned}$$

(The integrals above can be found by employing integration by parts.) Finally we can find the general solution for x:

$$\begin{aligned} x &= \Psi v \\ &= \begin{bmatrix} e^{2t} & -3e^t \\ 0 & e^t \end{bmatrix} \left(\begin{bmatrix} t - \frac{3}{2} te^{-2t} - \frac{3}{4} e^{-2t} \\ -te^{-t} - e^{-t} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right) \\ &= \begin{bmatrix} e^{2t} & -3e^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} t - \frac{3}{2} te^{-2t} - \frac{3}{4} e^{-2t} \\ -te^{-t} - e^{-t} \end{bmatrix} + \begin{bmatrix} e^{2t} & -3e^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} te^{2t} + \frac{3}{2} t + \frac{9}{4} \\ -t - 1 \end{bmatrix} + c_1 \begin{bmatrix} e^{2t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3e^t \\ e^t \end{bmatrix} \end{aligned}$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

Example: Consider the system

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

where a, b, c, d, k_1, k_2 are constants. Suppose that the coefficient matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has two distinct negative real eigenvalues. Show that the limits $\lim_{t \to +\infty} x_1(t)$ and $\lim_{t \to +\infty} x_2(t)$ exist and do not depend on the initial values of x_1 and x_2. Compute these limits in terms of A, k_1 and k_2.

Solution: Let the eigenvalues of A be λ_1 and λ_2. Since they are not equal to each other, the matrix A must be diagonalizable. So there exists an invertible matrix P (which we will not attempt to compute) such that

$$A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1}$$

Consequently, we have

$$\Psi(t) = Pe^{Jt} = P\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$\Psi^{-1}(t) = \begin{bmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{bmatrix} P^{-1}$$

In order to apply the variation of parameters formula, we will need to look at $\Psi^{-1}(t)\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{bmatrix} P^{-1}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$. The last product in this formula will again give us some vector of constants. So we can write

$$\Psi^{-1}(t)\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} e^{-\lambda_1 t} & 0 \\ 0 & e^{-\lambda_2 t} \end{bmatrix}\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} l_1 e^{-\lambda_1 t} \\ l_2 e^{-\lambda_2 t} \end{bmatrix}$$

for certain constants l_1, l_2. Now, let us apply the variation of parameters formula:

$$\begin{aligned} x &= \Psi(t)\int \Psi^{-1}(t)\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} dt \\ &= \Psi(t)\int \begin{bmatrix} l_1 e^{-\lambda_1 t} \\ l_2 e^{-\lambda_2 t} \end{bmatrix} dt \\ &= \Psi(t)\left(\begin{bmatrix} -\frac{l_1}{\lambda_1} e^{-\lambda_1 t} \\ -\frac{l_2}{\lambda_2} e^{-\lambda_2 t} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) \\ &= P\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\left(\begin{bmatrix} -\frac{l_1}{\lambda_1} e^{-\lambda_1 t} \\ -\frac{l_2}{\lambda_2} e^{-\lambda_2 t} \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right) \\ &= P\begin{bmatrix} -\frac{l_1}{\lambda_1} \\ -\frac{l_2}{\lambda_2} \end{bmatrix} + P\begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}. \end{aligned}$$

When t tends to infinity, the second summand above goes to 0 since both $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are decaying exponentials by assumption. The first summand is a constant. Therefore, the limit exists and it is independent of the initial values because it is independent of the values of the constants c_1, c_2.

In order to compute the limiting values $x_1(\infty)$ and $x_2(\infty)$, notice that the derivatives of the functions x_1 and x_2 will tend to 0 at infinity (to see this, we may for instance use the formula for x obtained above). Therefore, by considering the original system of differential equations, we must have

$$0 = A\begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} = -A^{-1}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$