Quantifiers and Quantified Statements

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Quantifiers

The logical system constructed so far is called **Propositional Logic**, where we have propositions and their combinations. In order to develop a logical system for mathematics, we need to incorporate variables into our system.

The basic idea is as follows:

- We have an alphabet for our system.
- Using symbols from the alphabet, we can create statements.
- e.g., $(2+2=5) \vee (2<4+y) \to x^2+y^2=z^2$
- We incorporate two quantifiers that quantify variables.

Notation

- $\exists$: **Existential Quantifier** (there exists / for some)
- $\forall$: **Universal Quantifier** (for all / for any / for every)

The intuitive meaning is as follows:

- $\forall x \in U, P(x)$ means that the statement $P(x)$ is true for **every possible choice** of x in the universe of discourse.
- $\exists x \in U, P(x)$ means that the statement $P(x)$ is true for **at least one choice** of x in the universe of discourse.

Free vs. Bound Variables

A variable in a statement that is not bound by a quantifier is called a **free variable**. A variable bound by a quantifier is called a **bound variable**.

Example:

$$\exists y \in \mathbb{R}, \quad x^2 + y = z$$

- y: Bound variable (bound by $\exists$)
- x: Free variable
- z: Free variable

Notations

$P(x_1, x_2, \ldots, x_n)$: A statement involving variables.

How to read quantified statements:

$$\forall x \in U, P(x)$$

"$P(x)$ holds for every x in U."

$$\exists x \in U, P(x)$$

"There exists x in U such that $P(x)$ holds."

Example of binding:

$$P(x, y) : \exists w \in \mathbb{N}, \quad x^2 + w^2 = y^3$$

Here, x and y are **free variables**, while w is a **bound variable**.

Examples: Expressing Statements

Example 1: Express the following statements in English, assuming that the universe is all people and $L(x, y)$ denotes "x loves y".

1. $\forall x \forall y, L(x, y)$: Everyone loves everyone.
2. $\forall x \exists y, L(x, y)$: Everyone loves someone.
3. $\exists y \forall x, L(x, y)$: There is someone who is loved by everyone.
4. $\exists x \forall y, L(x, y)$: There is someone who loves everyone.
5. $\forall y \exists x, L(x, y)$: Everyone is loved by someone.
6. $\exists x \exists y, L(x, y)$: Someone loves someone.
7. $\exists y \exists x, L(x, y)$: Someone loves someone.

Example 2: Express the following statements symbolically in formal logic using quantifiers, logical connectors, arithmetic symbols, etc.

Statement: "Every integer is divisible by any natural number which is greater than 1."

Let the universe be integers ($\mathbb{Z}$).

$$\forall x \in \mathbb{Z}, \quad \forall y \in \mathbb{N}, \quad (y > 1) \to \exists z \in \mathbb{Z}, (x = y \cdot z)$$

Alternatively, written using the divides relation ($|$):

$$\forall x \in \mathbb{Z}, \quad \forall y \in \mathbb{N}, \quad (y > 1 \to y|x)$$

Example 3: Statement: "There exists an integer that divides every integer greater than 1."

$$\exists x \in \mathbb{Z}, \quad \forall y \in \mathbb{Z}, \quad (y > 1 \to x|y)$$

Logical Implications

Some logical implications between quantified formulas:

1. $\forall x P(x) \implies \exists x P(x)$ (Assuming that the universe is not empty).
2. $\forall x \forall y P(x,y) \iff \forall y \forall x P(x,y)$ (We can swap universal quantifiers).
3. $\exists x \exists y P(x,y) \iff \exists y \exists x P(x,y)$ (We can swap existential quantifiers).
4. $\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$ (Note: The reverse is NOT generally true. "There is a key that opens every door" implies "Every door has a key that opens it", but not vise versa).

De Morgan's Laws for Quantifiers

Just as we have De Morgan's laws for logical connectives, we have them for quantifiers.

- $\neg(\forall x \in U, P(x)) \iff \exists x \in U, \neg P(x)$
- $\neg(\exists x \in U, P(x)) \iff \forall x \in U, \neg P(x)$

Example 1: For all primes P, assume that P has property $Q(x)$:

$$\forall x \in \mathbb{P}, Q(x)$$

The negation is:

$$\neg(\forall x \in \mathbb{P}, Q(x)) \iff \exists x \in \mathbb{P}, \neg Q(x)$$

Example 2 (Complex Negation): Consider the statement:

$$\forall x \in \mathbb{Z}, (P(x) \to Q(x))$$

Its negation is:

$$\neg(\forall x \in \mathbb{Z}, (P(x) \to Q(x))) \iff \exists x \in \mathbb{Z}, \neg(P(x) \to Q(x))$$

Using the equivalence $\neg(A \to B) \equiv A \land \neg B$, we get:

$$\exists x \in \mathbb{Z}, (P(x) \land \neg Q(x))$$

Valid Arguments with Quantifiers

How to prove equivalences or implications involving quantifiers? We use **Instantiation** and **Generalization** rules.

1. Universal Instantiation (UI)

$$\frac{\forall x \in U, P(x)}{P(c)}$$

Explanation: If $\forall x \in U, P(x)$ is true, then $P(c)$ is true for any c in the universe U.

- where c is any object in U.
- c can be arbitrary or specific.

2. Universal Generalization (UG)

$$\frac{P(c)}{\forall x \in U, P(x)}$$

Explanation: If we can prove $P(c)$ for an **arbitrary** element $c \in U$ (with no special properties assumed other than being in U), then we can generalize that $\forall x \in U, P(x)$.

- where c is an arbitrary object in U.

3. Existential Instantiation (EI)

$$\frac{\exists x \in U, P(x)}{P(b)}$$

Explanation: If we know $\exists x \in U, P(x)$, we can give a name to that element.

- where b is some particular object in U.
- **Important Condition:** b must be a **new** variable name not used before in the proof.

4. Existential Generalization (EG)

$$\frac{P(d)}{\exists x \in U, P(x)}$$

Explanation: If we show $P(d)$ is true for some object d, we can conclude that there exists an x such that $P(x)$.

- where d is some object in U.

Example of Instantiation

Proposition: Suppose that there exists 2 integers such that their squares sum up to 5. Formally:

$$\exists x \in \mathbb{Z}, \exists y \in \mathbb{Z}, (x^2 + y^2 = 5)$$

Using these addition rules, let:

1. Let's pick $a, b \in \mathbb{Z}$. (Instantiation)
2. Such that their squares sum up to 5.
$$a^2 + b^2 = 5$$
3. Now, using these definitions and rules, let $a^2 + b^2 = 5$ be true.
4. See an example of a valid argument:
$$a = 1, b = 2 \implies 1^2 + 2^2 = 1 + 4 = 5.$$
5. Since $P(1, 2)$ is true, then $\exists x \exists y P(x, y)$ is true (EG).

Example

Statements:

- Every cat that is nice and smart likes tuna.
- Every Siamese cat is nice.
- There exists a Siamese cat that is not smart.
- (Implicit context: There exists a Siamese cat that does not like tuna? Or checking validity?)

Let's formalize this argument.
Definitions:

- $N(x)$: x is nice
- $S(x)$: x is smart
- $T(x)$: x likes tuna
- $Si(x)$: x is Siamese
- U: The set of cats

Premises:

1. $\forall x \in U, ((N(x) \wedge S(x)) \to T(x))$
2. $\forall x \in U, (Si(x) \to N(x))$
3. $\exists x \in U, (Si(x) \wedge \neg T(x))$

Goal: Show that $\exists x \in U, \neg S(x)$ (There exists a cat that is not smart).

Derivation

(4) $Si(a) \wedge \neg T(a)$ by (3) and EI (Existential Instantiation)

(5) $(N(a) \wedge S(a)) \to T(a)$ by (1) and UI (Universal Instantiation)

(6) $\neg T(a)$ by (4) and Simplification

(7) $\neg T(a) \to \neg(N(a) \wedge S(a))$ by (5) and Contrapositive

(8) $\neg(N(a) \wedge S(a))$ by (6), (7) and Modus Ponens

(9) $\neg N(a) \vee \neg S(a)$ by (8) and De Morgan

(10) $Si(a) \to N(a)$ by (2) and UI

(11) $Si(a)$ by (4) and Simplification

(12) $N(a)$ by (10), (11) and Modus Ponens

(13) $\neg(\neg N(a))$ (Double Negation of 12)

(14) $\neg S(a)$ by (9), (12) and Disjunctive Syllogism ($A \vee B, \neg A \vdash B$)

(15) $\exists x \in U, \neg S(x)$ by (14) and EG (Existential Generalization)

Example: Derivation with Quantifiers

Problem: Find a derivation for the following valid argument.

Premises:

1. $\forall x \in W, \exists y \in W, (E(x) \to M(x) \vee N(y))$
2. $\neg(\forall x \in W, M(x))$
3. $\forall x \in W, E(x)$

Conclusion:

$$\exists x \in W, N(x)$$

Solution (Derivation)

(4) $\exists x \in W, \neg M(x)$ — by (2) and Quantifier Negation (De Morgan)

(5) $\neg M(a)$ — by (4) and EI (Existential Instantiation)

(6) $E(a)$ — by (3) and UI (Universal Instantiation)

(7) $\exists y \in W, (E(a) \to M(a) \vee N(y))$ — by (1) and UI (instantiating x with a)

(8) $E(a) \to (M(a) \vee N(b))$ — by (7) and EI (instantiating y with b)

(9) $M(a) \vee N(b)$ — by (6), (8) and Modus Ponens

(10) $N(b)$ — by (5), (9) and Disjunctive Syllogism

(11) $\exists x \in W, N(x)$ — by (10) and EG (Existential Generalization)