Relationship between statements

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Logical Implication

Given two statements p and q, we say that p implies q and write $p \Rightarrow q$ in the case that q is true in all cases that p is true; equivalently the statement $p \rightarrow q$ is a tautology.

Theorem

Let P, Q, R, S be statements, then:

1. $((p \rightarrow q) \wedge p) \Rightarrow q$ (Modus Ponens)
2. $(p \rightarrow q) \wedge \neg q \Rightarrow \neg p$ (Modus Tollens)
3. $p \wedge q \Rightarrow p$
4. $p \wedge q \Rightarrow q$
5. $p \leftrightarrow q \Rightarrow q \rightarrow p$
6. $(p \rightarrow q) \wedge (q \rightarrow p) \Rightarrow p \leftrightarrow q$
7. $q \Rightarrow p \vee q$
8. $(p \vee q) \wedge \neg p \Rightarrow q$
9. $(p \vee q) \wedge \neg q \Rightarrow p$
10. $p \leftrightarrow q \Rightarrow p \rightarrow q$
11. $(p \rightarrow q) \wedge (R \rightarrow S) \wedge (p \vee R) \Rightarrow (q \vee S)$
12. $(p \rightarrow q) \wedge (q \rightarrow R) \Rightarrow p \rightarrow R$ (Hypothetical Syllogism)

Proof of $(p \vee q) \wedge \neg p \Rightarrow q$

We construct the truth table of the following statement: $(p \vee q) \wedge \neg p \rightarrow q$

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p \rightarrow q$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

(We use truth tables to prove a tautology)

Logical Equivalence

Given two statements P and Q, we say that P is equivalent to Q and we write $P \Leftrightarrow Q$ in the case that $P \leftrightarrow Q$ is a tautology.

Theorem

Let P, Q, R be statements, Then:

a) $\neg(\neg p) \Leftrightarrow p$ (Double Negation)

b) $p \vee q \Leftrightarrow q \vee p$ (Commutative Law)

c) $p \wedge q \Leftrightarrow q \wedge p$ (Commutative Law)

d) $(p \vee q) \vee R \Leftrightarrow p \vee (q \vee R)$ (Associative Law)

e) $(p \wedge q) \wedge R \Leftrightarrow p \wedge (q \wedge R)$ (Associative Law)

f) $p \wedge (q \vee R) \Leftrightarrow (p \wedge q) \vee (p \wedge R)$ (Distributive Law)

g) $p \vee (q \wedge R) \Leftrightarrow (p \vee q) \wedge (p \vee R)$ (Distributive Law)

h) $p \rightarrow q \Leftrightarrow \neg p \vee q$ (Implication Law)

i) $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$ (Contrapositive)

j) $p \Leftrightarrow q \Leftrightarrow q \Leftrightarrow p$

k) $p \Leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

l) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ (De Morgan's Law)

m) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ (De Morgan's Law)

n) $\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$

o) $\neg(p \leftrightarrow q) \Leftrightarrow (p \wedge \neg q) \vee (q \wedge \neg p)$

Proof of $(p \rightarrow q) \Leftrightarrow \neg p \vee q$

We have to construct a truth table to prove that it is a tautology. The statement is: $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

p	q	$p \rightarrow q$	$\neg p \vee q$	$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Hence, $(p \rightarrow q) \Leftrightarrow \neg p \vee q$ is a tautology.

Proof of part (n) without truth tables

We want to show: $\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$

$$\begin{aligned}
\neg(p \rightarrow q) &\Leftrightarrow \neg(\neg p \vee q) && \text{(by h: } p \rightarrow q \Leftrightarrow \neg p \vee q\text{)} \\
&\Leftrightarrow \neg(\neg p) \wedge \neg q && \text{(by m: De Morgan's)} \\
&\Leftrightarrow p \wedge \neg q && \text{(by a: Double Negation)}
\end{aligned}$$

Variations of Conditional Statement

Given a statement of the form $p \to q$, the statement:

- $\neg q \to \neg p$ is called the **contrapositive** of $p \to q$.
- $q \to p$ is called the **converse** of $p \to q$.
- $\neg p \to \neg q$ is called the **inverse** of $p \to q$.

Example

Show that: $(A \wedge B) \to Q \Leftrightarrow A \to (B \to Q)$

1st Method

By definition, we show that $((A \wedge B) \to Q) \leftrightarrow (A \to (B \to Q))$ is a tautology by constructing the truth table.

2nd Method

We can use the known equivalences between various statements to get the wanted equivalence.

$$\begin{aligned}
(A \wedge B) \to Q &\Leftrightarrow \neg(A \wedge B) \vee Q && \text{(because } p \to q \Leftrightarrow \neg p \vee q) \\
&\Leftrightarrow (\neg A \vee \neg B) \vee Q && \text{(because } \neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q) \\
&\Leftrightarrow \neg A \vee (\neg B \vee Q) && \text{(because } (p \vee q) \vee R \Leftrightarrow p \vee (q \vee R)) \\
&\Leftrightarrow \neg A \vee (B \to Q) && \text{(because } \neg p \vee q \Leftrightarrow p \to q) \\
&\Leftrightarrow A \to (B \to Q) && \text{(because } \neg p \vee q \Leftrightarrow p \to q)
\end{aligned}$$