

**Example(1)** Prove by induction that the following equality holds for all integers  $k \geq 1$ :

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2k-1) \cdot (2k+1)} = \frac{k}{2k+1}.$$

**Solution:**

We will do a proof by induction.

**Induction Basis:** Observe that, when we consider the base case, i.e.,  $k = 1$ , we have that

$$\frac{1}{1 \cdot 3} = \frac{1}{3} = \frac{1}{2 \cdot 1 + 1}$$

Therefore, the statement is satisfied for the base case,

**Inductive Step:** Assume, for a fixed but an arbitrary  $n \in \{a \in \mathbb{N} \mid a \geq 1\}$ , that we have  $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1) \cdot (2n+1)} = \frac{n}{2n+1}$ . (Induction Hypothesis/Assumption)

We wish to show that the statement is also true for  $n + 1$ .

Now, observe that, by induction assumption, we have

$$\begin{aligned} \frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2(n+1)-1) \cdot (2(n+1)+1)} &= \frac{n}{2n+1} + \frac{1}{(2(n+1)-1) \cdot (2(n+1)+1)} \\ &= \frac{n}{2n+1} + \frac{1}{(2n+1) \cdot (2n+3)} \\ &= \frac{n \cdot (2n+3)}{(2n+1) \cdot (2n+3)} + \frac{1}{(2n+1) \cdot (2n+3)} \\ &= \frac{2n^2 + 3n + 1}{(2n+1) \cdot (2n+3)} \\ &= \frac{(2n+1) \cdot (n+1)}{(2n+1) \cdot (2n+3)} \\ &= \frac{n+1}{2n+3} \\ &= \frac{n+1}{2(n+1)+1}. \end{aligned}$$

Therefore, the statement also holds for  $n + 1$ .

Hence, by PMI(Principle of Mathematical Induction), the theorem follows.

**Example(2)** Using induction, prove that

$$2^n + n^2 < 3^n$$

for all integers  $n \geq 4$ .

**Solution:**

We will do a proof by induction.

**Induction Basis:** Observe that, for the base case, i.e.,  $n=4$ , we have

$$(2^n + n^2)|_{n=4} = 2^4 + 4^2 = 32 \wedge 3^n|_{n=4} = 3^4 = 81.$$

We know that  $81 > 32$  and hence, for the base case, the statement holds.

**Inductive Step:** Assume, for a fixed but an arbitrary  $k \in \{a \in \mathbb{N} \mid a \geq 4\}$ , that we have  $2^k + k^2 < 3^k$ . (Induction Hypothesis/Assumption) We wish to show that the statement also holds for  $k + 1$ .

Now, by induction assumption, we have

$$2^{k+1} + (k+1)^2 = 2 \cdot 2^k + k^2 + 2k + 1 = (2^k + k^2) + 2^k + 2k + 1 < 3^k + 2^k + 2k + 1.$$

Here, we can see that  $3^{k+1} = 3^k \cdot 3 = 3^k + 3^k + 3^k = 3^k + 3^k \cdot 2$ . Furthermore, we notice that  $2^k < 3^k$ , as  $k$  is a natural number and  $3 > 2$ . Also,  $3^k > 2k + 1$ . We note that this fact is obvious and will not be proven here, but it can be shown utilizing some calculus and initial condition  $k = 4$ .

Now, we have  $3^k > 2^k \wedge 3^k > 2k + 1 \implies 2 \cdot 3^k > 2^k + 2k + 1$ . It is finally followed by

$$3^k + 2^k + 2k + 1 < 3^k + 2 \cdot 3^k = 3 \cdot 3^k = 3^{k+1}$$

and so

$$2^{k+1} + (k+1)^2 < 3^{k+1}.$$

Hence, by PMI(Principle of Mathematical Induction), the theorem follows.